

# Multipartite Quantum Correlation and Communication Complexities

Rahul Jain<sup>\*</sup>      Zhaohui Wei<sup>†</sup>      Penghui Yao<sup>‡</sup>      Shengyu Zhang<sup>§</sup>

## Abstract

The concepts of quantum correlation complexity and quantum communication complexity were recently proposed to quantify the minimum amount of resources needed in generating bipartite classical or quantum states in the single-shot setting. The former is the minimum size of the initially shared state  $\sigma$  on which local operations by the two parties (without communication) can generate the target state  $\rho$ , and the latter is the minimum amount of communication needed when initially sharing nothing. In this paper, we generalize these two concepts to multipartite cases, for both exact and approximate state generation. Our results are summarized as follows.

1. For multipartite pure states, the correlation complexity can be completely characterized by local ranks of subsystems.
2. We extend the notion of PSD-rank of matrices to that of tensors, and use it to bound the quantum correlation complexity for generating multipartite classical distributions.
3. For generating multipartite mixed quantum states, communication complexity is not always equal to correlation complexity (as opposed to bipartite case). But they differ by at most a factor of 2. Generating a multipartite mixed quantum state has the same communication complexity as generating its optimal purification. But for correlation complexity of these two tasks can be different (though still related by less than a factor of 2).
4. To generate a bipartite classical distribution  $P(x, y)$  approximately, the quantum communication complexity is completely characterized by the approximate PSD-rank of  $P$ . The quantum correlation complexity of approximately generating multipartite pure states is bounded by approximate local ranks.

## 1 Introduction

Shared randomness and quantum entanglement among parties located at different places are important resources for various distributed information processing tasks. How to generate these shared resources has been one of the most important issues, and recently much attention has been paid to the minimum amount of shared correlation and communication needed to generate bipartite classical and quantum states in one-shot setting [1, 5, 4, 12, 6]. In particular, in [6] the worst-case

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<sup>\*</sup>Department of Computer Science and Centre for Quantum Technologies, National University of Singapore, Singapore. Email: rahul@comp.nus.edu.sg

<sup>†</sup>School of Physics and Mathematical Sciences, Nanyang Technological University and Centre for Quantum Technologies, Singapore. Email: weizhaohui@gmail.com

<sup>‡</sup>Centre for Quantum Technologies, National University of Singapore, Singapore. Email: phyao1985@gmail.com

<sup>§</sup>Department of Computer Science and Engineering, and The Institute of Theoretical Computer Science and Communications, The Chinese University of Hong Kong. Email: syzhang@cse.cuhk.edu.hk

costs of several single-shot bipartite schemes to generate correlations and quantum entanglement have been characterized. The setting is as follows. Suppose that two parties, Alice and Bob, need to generate correlated random variables  $X$  and  $Y$ , with Alice outputting  $X$  and Bob outputting  $Y$ , such that  $(X, Y)$  is distributed according to a target distribution  $P$ . If  $P$  is not a product distribution, Alice and Bob could generate  $P$  by sharing an initial seed distribution  $(X', Y')$ , Alice owning  $X'$  and Bob owning  $Y'$ , and then each performing local operations on their own part. The minimal size of this seed correlation  $(X', Y')$  is defined as *randomized correlation complexity* [12], denoted  $R(P)$ , where the *size* of a bipartite distribution is defined as the half of the total number of bits. It has been known that  $R(P)$  is fully characterized as  $\lceil \log_2 \text{rank}_+(P) \rceil$  [12], where  $\text{rank}_+(P)$  is the nonnegative rank of matrix  $P^1$ , a measure in linear algebra with numerous applications in combinatorial optimization [11], nondeterministic communication complexity [7], algebraic complexity theory [9], and many other fields [2]. The problem becomes even more interesting when quantum operations are allowed: Alice and Bob share a seed quantum state  $\sigma$  and perform local quantum operations to generate a distributed classical distribution  $P$ . In this case, the minimal size of the seed quantum state  $\sigma$  is defined as *quantum correlation complexity*, denoted  $\text{QCorr}(P)$ , where the *size* of a bipartite quantum state is the half of the total number of qubits. One of the main results of [6] is that  $\text{QCorr}(P)$  could be completely characterized as  $\lceil \log_2 \text{rank}_{\text{psd}}(P) \rceil$ , where  $\text{rank}_{\text{psd}}(P)$  is the PSD-rank of  $P$ , a concept recently proposed by Fiorini *et al.* in studies of the minimum size of extended formulations of optimization problems such as TSP [4]. Since  $\text{rank}_+(P)$  could be much larger than  $\text{rank}_{\text{psd}}(P)$ , this implies a potentially huge advantage of using quantum operations, over classical counterparts, to generate classical distributions.

More generally, the target state can be a quantum state  $\rho$ , and [6] gave a complete characterization for the minimum size of the seed state to generate  $\rho$ . In particular, if  $\rho$  is a pure state  $|\psi\rangle\langle\psi|$  and an  $\epsilon$ -approximation is allowed for generating  $|\psi\rangle\langle\psi|$ , then the correlation complexity is completely characterized by the  $(1 - \epsilon)^2$ -cutoff point of the Schmidt coefficients of  $|\psi\rangle$ , closing a possibly exponential gap left in [1].

The above discussion assumes that Alice and Bob perform local operations on a shared state. Actually, Alice and Bob could replace the shared states discussed above by communication. In this case, the minimal amount of communication in classical and quantum protocols for generating target classical distribution  $P$  are defined as the *randomized* and *quantum communication complexities*, denoted by  $\text{RComm}(P)$  and  $\text{QComm}(P)$ , respectively [12]; one can similarly define  $\text{QComm}(\rho)$  for generating quantum states  $\rho$ . We have introduced correlation complexity and communication complexity. An interesting fact is that when only two parties are involved, these two measures are always the same, and this is true for both classical and quantum settings [12].

Capturing the minimum cost to generate target states, the concepts of correlation complexity and communication complexity are fundamental parameters of the shared states as a resource. In particular, when the target state is quantum, the resource is entanglement, arguably the most important shared resource in almost all quantum information processing tasks. While bipartite entanglement is well understood, multipartite entanglement has been elusive on many levels, and considerable efforts have been made to study it from various angles. In this paper, we extend the study of correlation and communication complexity of generating a classical correlation and quantum entanglement to multipartite cases. Our results are summarized next, and we hope that they can shed light on multipartite entanglement from another fundamental perspective.

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<sup>1</sup>A bipartite distribution  $P$  is also natural a matrix  $[P(x, y)]_{x, y}$ , and we thus use  $P$  for both the distribution and the matrix.

## 1.1 Multipartite quantum correlation complexity

For multipartite cases, it turns out that quantum correlation complexity and quantum communication complexity are not equivalent any more, thus we have to deal with them separately. We first consider quantum correlation complexity of generating a  $k$ -partite state.

**Definition 1.** Suppose  $k$  parties,  $A_1, A_2, \dots, A_k$ , share a seed state  $\sigma$ , and they aim to generate a target state  $\rho$  by each perform some operation on her own part of  $\sigma$ . Then the quantum correlation complexity of  $\rho$ , denoted  $\text{QCorr}(\rho)$ , is the minimal size of  $\sigma$  such that local quantum operations on  $\sigma$  can generate  $\rho$ . Here the size of  $\sigma$  is defined as  $\sum_{i=1}^k n_i$ , where  $n_i$  is the number of qubits of  $\sigma$  held by  $A_i$ .

Let us first consider the case of  $\rho$  being a pure state. For a bipartite pure state  $|\psi\rangle$ , Schmidt decompositions help us to characterize  $\text{QCorr}(|\psi\rangle)$  and  $\text{QComm}(|\psi\rangle)$  perfectly, but multipartite pure states do not have Schmidt decompositions in general. It turns out that the quantum correlation complexity is the sum of the “marginal complexity”.

**Definition 2.** Suppose  $|\psi\rangle$  is a pure state in  $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_k$ , and  $\rho_j$  is the reduced density matrices of  $|\psi\rangle$  in  $\mathcal{H}_j$ . Define the marginal complexity of  $|\psi\rangle$  as

$$M(|\psi\rangle) = \sum_{j=1}^k \lceil \log_2 \text{rank}(\rho_j) \rceil.$$

**Theorem 1.** Suppose  $|\psi\rangle$  is a pure state in  $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_k$ , and  $\rho_j$  is the reduced density matrices of  $|\psi\rangle$  in  $\mathcal{H}_j$ . Then

$$\text{QCorr}(|\psi\rangle) = M(|\psi\rangle).$$

For a mixed quantum state  $\rho$ , however, the correlation complexity is less clear. It was mentioned that in the bipartite case,  $\text{QCorr}(\rho)$  is exactly the minimal  $\text{QCorr}(|\psi\rangle)$  over all purifications  $|\psi\rangle$  of  $\rho$  [6]. This turns out to be not the case any more in multipartite setting.

**Theorem 2.** Assume that  $\rho$  is a quantum state in  $\bigotimes_{i=1}^k \mathcal{H}_i$ . Then we have

$$\text{QCorr}(\rho) \leq r(\rho) \leq \left(2 - \frac{2}{k}\right) \text{QCorr}(\rho),$$

where  $r(\rho)$  is the minimum  $\text{QCorr}(|\psi\rangle)$  over all purifications  $|\psi\rangle$  of  $\rho$ .

We will also show that both inequalities in the above theorem are tight, thereby implying that  $\text{QCorr}(\rho)$  is indeed different from  $\min\{\text{QCorr}(|\psi\rangle) : |\psi\rangle \text{ purifies } \rho\}$ .

While in some sense pure quantum states contain the most “quantumness” in terms of superposition, the other extreme is mixture of classical states, *i.e.*, classical distributions. In the bipartite case, the quantum correlation complexity of generating distribution  $P = [P(x, y)]_{x, y}$  is exactly  $\lceil \log_2 \text{rank}_{\text{psd}}(P) \rceil$ . We will show an analogous result in multipartite cases. To this end, we need to first generalize the notion of PSD-rank from matrices to tensors. Similar to the bipartite case, a  $k$ -partite probability distribution  $P = [P(x_1, x_2, \dots, x_k)]_{x_1, x_2, \dots, x_k}$  can also be viewed as a tensor of dimension  $k$ .

**Definition 3.** For an entry-wise nonnegative tensor  $P = [P(x_1, \dots, x_k)]_{x_1, \dots, x_k}$  of dimension  $k$ , its PSD-rank  $\text{rank}_{\text{psd}}^{(k)}(P)$  is the minimum  $r$  s.t. there are  $r \times r$  PSD matrices  $C_{x_1}^{(1)}, \dots, C_{x_k}^{(k)} \succeq 0$  with  $P(x_1, \dots, x_k) = \sum_{i,j=1}^r C_{x_1}^{(1)}(i, j) \cdots C_{x_k}^{(k)}(i, j)$ .

With this definition, we can bound the quantum correlation complexity of  $P$  in terms of its PSD-rank.

**Theorem 3.** Suppose  $P = [P(x_1, \dots, x_k)]_{x_1, \dots, x_k}$  is a probability distribution on  $\mathcal{X}_1 \times \cdots \times \mathcal{X}_k$ . Then

$$\frac{k}{2k-2} \lceil \log_2 \text{rank}_{\text{psd}}^{(k)}(P) \rceil \leq \text{QCorr}(P) \leq k \lceil \log_2 \text{rank}_{\text{psd}}^{(k)}(P) \rceil.$$

## 1.2 Multipartite quantum communication complexity

As earlier mentioned, one can also consider the setting in which the players share nothing at the beginning and communicate to generate some target state. The communication complexity is formally defined as follows.

**Definition 4.** Suppose  $k$  parties,  $A_1, A_2, \dots, A_k$ , initially share nothing and aim to jointly generate a quantum state  $\rho$  by communication. The quantum communication complexity of generating  $\rho$ , denoted  $\text{QComm}(\rho)$ , is the minimum number of qubits exchanged between these  $k$  parties.

The following theorem gives bounds for quantum communication complexity of pure states. Recall that  $M(|\psi\rangle) = \sum_{j=1}^k \lceil \log_2 \text{rank}(\rho_i) \rceil$ , where  $\rho_i$  is  $|\psi\rangle$  reduced to Player  $i$ 's space.

**Theorem 4.** Suppose  $|\psi\rangle$  is a  $k$ -partite pure state. Then

$$\frac{1}{2} M(|\psi\rangle) \leq \text{QComm}(|\psi\rangle) \leq \frac{k-1}{k} M(|\psi\rangle).$$

Next we turn to general multipartite quantum mixed states. Different than quantum correlation complexity, the quantum communication complexity  $\text{QComm}(\rho)$  is always equal to the minimum  $\text{QComm}(|\psi\rangle)$  over purifications  $|\psi\rangle$  of  $\rho$ .

**Theorem 5.** For any  $k$ -partite quantum state  $\rho$ ,

$$\text{QComm}(\rho) = \min\{\text{QComm}(|\psi\rangle) : |\psi\rangle \text{ is a purification of } \rho\}.$$

Combining the results in the above two subsections together, we get the following relationship between  $\text{QCorr}(\rho)$  and  $\text{QComm}(\rho)$  for a general multipartite quantum state  $\rho$ .

**Corollary 6.** For any  $k$ -partite quantum state  $\rho$ ,

$$\frac{k}{k-1} \text{QComm}(\rho) \leq \text{QCorr}(\rho) \leq 2 \text{QComm}(\rho).$$

## 1.3 Approximate quantum correlation complexity

In this section, we consider relaxing the task of state generation by allowing approximation. After all, we usually generate the state for some later information processing purpose, and thus if the generated state  $\rho'$  is very close to the target state  $\rho$ , then the same precision can be preserved after whatever further operations, global or local.

### 1.3.1 Bipartite

When a good approximation instead of the exact generation is satisfactory, the minimum size of the shared seed state can be smaller than that for the exact generation. In [6], a natural definition for the approximate correlation complexity was given as follows.

**Definition 5.** Let  $\rho$  be a bipartite quantum state in  $\mathcal{H}_A \otimes \mathcal{H}_B$ , and  $\epsilon > 0$ . Define

$$\text{QCorr}_\epsilon(\rho) \stackrel{\text{def}}{=} \min\{\text{QCorr}(\rho') : \rho' \in \mathcal{H}_A \otimes \mathcal{H}_B \text{ and } F(\rho, \rho') \geq 1 - \epsilon\}$$

Since for any bipartite state  $\rho$ , generating a mixed state is the same as generating its (optimal) purification [12, 6]

$$\text{QCorr}(\rho) = \min\{\text{QCorr}(|\psi\rangle) : |\psi\rangle \text{ purifies } \rho\} = \min\{\lceil \log_2 \mathbf{S}\text{-rank}(|\psi\rangle) \rceil : |\psi\rangle \text{ purifies } \rho\},$$

it is also natural to give another definition by putting the approximation on the purification instead of the original target state. Let

$$\text{QCorr}'_\epsilon(\rho) = \min\{\text{QCorr}(|\psi'\rangle) : |\psi'\rangle \text{ purifies } \rho, F(|\psi\rangle, |\psi'\rangle) \geq 1 - \epsilon\}.$$

As we will show, these two definitions are equivalent, *i.e.*,  $\text{QCorr}_\epsilon(\rho) = \text{QCorr}'_\epsilon(\rho)$ . Note that the second definition is easier to analyze, since the approximate correlation complexity for pure states are well understood [6]: For any  $|\psi\rangle = \sum_{x,y} A(x,y)|x\rangle|y\rangle$ , let matrix  $A = [A(x,y)]$ , then

$$\min\{\text{QCorr}(|\psi'\rangle) : F(|\psi\rangle, |\psi'\rangle) \geq 1 - \epsilon\} = \mathbf{rank}_{2\epsilon-\epsilon^2}(A),$$

where the approximate rank  $\mathbf{rank}_\delta(A)$  of a matrix  $A$  is the smallest number  $r$  *s.t.* the summation of the largest  $r$  singular values squared is at least  $1 - \delta$ .

Based on this result, we could get the following characterization of  $\text{QCorr}_\epsilon(\rho)$  for the special case of classical  $\rho$ , namely when  $\rho$  is a classical distribution  $P$ . We first define approximate PSD-rank and approximate correlation complexity by classical states as follows.

**Definition 6.**  $P = [p(x,y)]_{x,y}$  is a bipartite probability distribution, its  $\epsilon$ -approximate PSD-rank is

$$\mathbf{rank}_{\text{psd},\epsilon}(P) = \min\{\mathbf{rank}_{\text{psd}}(P') : F(P, P') \geq 1 - \epsilon\}. \quad (1)$$

where  $P'$  is another probability distribution on the same sample space of  $P$ .

**Definition 7.** For a bipartite classical distribution  $P = [P(x,y)]_{x,y}$ , its  $\epsilon$ -approximate quantum correlation complexity by classical state is  $\text{QCorr}_\epsilon^{\text{cla}}(P) = \min\{\text{QCorr}(P') : F(P, P') \geq 1 - \epsilon\}$ , where  $P'$  is another probability distribution on the same sample space of  $P$ .

The following theorem says that the most efficient approximate generation of a classical state can always be achieved by another classical state. Moreover, the approximate correlation complexity of a classical state could be completely characterized by the approximate PSD-rank.

**Theorem 7.** For any classical state  $P = [P(x,y)]_{x,y}$ ,

$$\text{QCorr}_\epsilon(P) = \text{QCorr}_\epsilon^{\text{cla}}(P) = \lceil \log_2 \mathbf{rank}_{\text{psd},\epsilon}(P) \rceil.$$

Finally, for the general case of an arbitrary quantum state  $\rho$ , we give the following characterization of  $\text{QCorr}_\epsilon(\rho)$ .

**Theorem 8.** *Let  $\sigma$  be an arbitrary quantum state in  $\mathcal{H}_A \otimes \mathcal{H}_B$ , and  $0 < \epsilon < 1$ . Then  $\text{QCorr}_\epsilon(\sigma) = \lceil \log_2 r \rceil$ , where  $r$  is the minimum integer s.t. there exist a collection of matrices,  $\{A_x\}$ 's and  $\{B_y\}$ 's of the same column number  $l \geq r$ , satisfying the following conditions.*

1. *The matrices relate to  $\sigma$  by the following equation.*

$$\sigma = \sum_{x,x';y,y'} |x\rangle\langle x'| \otimes |y\rangle\langle y'| \cdot \text{tr}\left((A_x^\dagger A_x)^T (B_{y'}^\dagger B_{y'})\right). \quad (2)$$

2. *Denoting the  $i$ -th column of any matrix  $M$  by  $|M(i)\rangle$ , then*

$$\sum_x \langle A_x(i) | A_x(j) \rangle = \sum_y \langle B_y(i) | B_y(j) \rangle = 0, \quad (3)$$

3.

$$\sum_{i=1}^r \left( \sum_x \langle A_x(i) | A_x(i) \rangle \right) \left( \sum_y \langle B_y(i) | B_y(i) \rangle \right) \geq 1 - \epsilon, \quad (4)$$

### 1.3.2 Multipartite

As in [6], it is natural to consider two different approximations to a pure target state, one to approximate by a mixed state, and the other to approximate by a pure state.

**Definition 8.** *Let  $\epsilon > 0$ . Let  $|\psi\rangle$  be a  $k$ -partite quantum pure state in  $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_k$ . Define*

$$\text{QCorr}_\epsilon(|\psi\rangle) \stackrel{\text{def}}{=} \min\{\text{QCorr}(\rho') : \rho' \text{ is in } \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_k \text{ and } F(|\psi\rangle\langle\psi|, \rho') \geq 1 - \epsilon\}$$

and

$$\text{QCorr}_\epsilon^{\text{pure}}(|\psi\rangle) \stackrel{\text{def}}{=} \min\{\text{QCorr}(|\phi\rangle) : |\phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_k \text{ and } F(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|) \geq 1 - \epsilon\}.$$

We can see that  $\text{QCorr}_\epsilon(\rho)$  and  $\text{QCorr}_\epsilon^{\text{pure}}(\rho)$  are the complexities of approximating  $\rho$  by mixed and pure states respectively.

For a  $k$ -partite pure state  $|\psi\rangle$  in  $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_k$ , let  $\rho_i$  be the reduced density matrix of  $|\psi\rangle$  in  $\mathcal{H}_i$ , and  $r_i = \text{rank}(\rho_i)$ . Denote the  $\epsilon$ -approximate Schmidt rank of  $|\psi\rangle$  with respect to the separation  $(A_i, A_{-i})$  (here  $A_{-i} = A_1 \dots A_{i-1} A_{i+1} \dots A_k$ ) as  $r_i^{(\epsilon)}$ , i.e.,  $r_i^{(\epsilon)} = \mathbf{S}\text{-rank}_\epsilon^{(A_i, A_{-i})}(|\psi\rangle)$ . Then we have

**Theorem 9.** *Let  $|\psi\rangle \in \bigotimes_{i=1}^k \mathcal{H}_i$  be a  $k$ -partite state,  $\epsilon > 0$ , and  $M_\epsilon(|\psi\rangle) = \sum_{i=1}^k \lceil \log_2 r_i^{(\epsilon)} \rceil$ . Then*

$$M_\epsilon(|\psi\rangle) \leq \text{QCorr}_\epsilon^{\text{pure}}(|\psi\rangle) \leq M_{\epsilon/k}(|\psi\rangle).$$

Finally, we consider the relationship between  $\text{QCorr}_\epsilon(|\psi\rangle)$  and  $\text{QCorr}_\epsilon^{\text{pure}}(|\psi\rangle)$ .

**Theorem 10.** *Let  $|\psi\rangle \in \mathcal{H}_{A_1} \otimes \dots \otimes \mathcal{H}_{A_k}$  be a pure state and  $\epsilon > 0$ . Then*

$$\frac{k}{2k-2} \text{QCorr}_{k\epsilon}^{\text{pure}}(|\psi\rangle) \leq \text{QCorr}_\epsilon(|\psi\rangle) \leq \text{QCorr}_\epsilon^{\text{pure}}(|\psi\rangle).$$

## 2 Preliminaries

In this paper we consider multipartite systems. If a system has  $k$  parties, we usually use  $A_1, \dots, A_k$  to denote them. Their spaces are  $\mathcal{H}_1, \dots, \mathcal{H}_k$ , respectively. For notational convenience, we use  $A_{-i}$  for  $A_1 \dots A_{i-1} A_{i+1} \dots A_k$ , and use subscript  $-i$  in other symbols (such as  $\mathcal{H}_{-i}$ ) for a similar meaning.

**Matrix theory.** For a natural number  $n$  we let  $[n]$  represent the set  $\{1, 2, \dots, n\}$ . We sometimes write  $A = [A(x, y)]$  to mean that  $A$  is a matrix with the  $(x, y)$ -th entry being  $A(x, y)$ . An operator  $A$  is said to be *Hermitian* if  $A^\dagger = A$ . A Hermitian operator  $A$  is said to be *positive semi-definite* (PSD) if all its eigenvalues are non-negative. For any vectors  $|v_1\rangle, \dots, |v_r\rangle$  in  $\mathbb{C}^n$ , the  $r \times r$  matrix  $M$  defined by  $M(i, j) \stackrel{\text{def}}{=} \langle v_i | v_j \rangle$  is positive semi-definite. The following definition of PSD-rank of a matrix was proposed in [4].

**Definition 9.** For a matrix  $P \in \mathbb{R}_+^{n \times m}$ , its PSD-rank, denoted  $\text{rank}_{\text{psd}}(P)$ , is the minimum number  $r$  such that there are PSD matrices  $C_x, D_y \in \mathbb{C}^{r \times r}$  with  $\text{tr}(C_x D_y) = P(x, y)$ ,  $\forall x \in [n], y \in [m]$ .

One can see that this corresponds to the special case of  $k = 2$  in Definition 3. When  $k = 2$ , we drop the superscript (2) in Definition 3, thus making it consistent with the above definition of PSD-rank of matrices.

**Quantum information.** A quantum state  $\rho$  in Hilbert space  $\mathcal{H}$ , denoted  $\rho \in \mathcal{H}$ , is a trace one positive semi-definite operator acting on  $\mathcal{H}$ . A quantum state  $\rho$  is called *pure* if it is rank one, namely  $\rho = |\psi\rangle\langle\psi|$  for some vector  $|\psi\rangle$  of unit  $\ell_2$  norm; in this case, we often identify  $\rho$  with  $|\psi\rangle$ . For quantum states  $\rho$  and  $\sigma$ , their fidelity is defined as  $F(\rho, \sigma) \stackrel{\text{def}}{=} \text{tr}(\sqrt{\sigma^{1/2} \rho \sigma^{1/2}})$ . For  $\rho, |\psi\rangle \in \mathcal{H}$ , we have  $F(\rho, |\psi\rangle\langle\psi|) = \sqrt{\langle\psi|\rho|\psi\rangle}$ . We define norm of  $|\psi\rangle$  as  $\| |\psi\rangle \| \stackrel{\text{def}}{=} \sqrt{\langle\psi|\psi\rangle}$ . For a quantum state  $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$ , we let  $\text{tr}_{\mathcal{H}_B} \rho$  represent the partial trace of  $\rho$  in  $\mathcal{H}_A$  after tracing out  $\mathcal{H}_B$ . Let  $\rho \in \mathcal{H}_A$  and  $|\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$  be such that  $\text{tr}_{\mathcal{H}_B} |\phi\rangle\langle\phi| = \rho$ , then we call  $|\phi\rangle$  a *purification* of  $\rho$ .

**Definition 10.** For a pure state  $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ , its Schmidt decomposition is defined as

$$|\psi\rangle = \sum_{i=1}^r \sqrt{p_i} \cdot |v_i\rangle \otimes |w_i\rangle,$$

where  $|v_i\rangle$ 's are orthonormal states in  $\mathcal{H}_A$ ,  $|w_i\rangle$ 's are orthonormal states in  $\mathcal{H}_B$ , and  $p$  is a probability distribution.

It is easily seen that  $r$  is also equal to  $\text{rank}(\text{tr}_{\mathcal{H}_A} |\psi\rangle\langle\psi|) = \text{rank}(\text{tr}_{\mathcal{H}_B} |\psi\rangle\langle\psi|)$  and is therefore the same in all Schmidt decompositions of  $|\psi\rangle$ . This number is also referred to as the *Schmidt rank* of  $|\psi\rangle$  and denoted  $\mathbf{S}\text{-rank}^{(A,B)}(|\psi\rangle)$ . The superscript  $(A, B)$  is to emphasize that the partition is between  $A$  and  $B$ . The next fact can be shown by considering Schmidt decomposition of the pure states involved; see, for example, Ex(2.81) of [8].

**Fact 11.** Let  $|\psi\rangle, |\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$  be such that  $\text{tr}_{\mathcal{H}_B} |\phi\rangle\langle\phi| = \text{tr}_{\mathcal{H}_B} |\psi\rangle\langle\psi|$ . There exists a unitary operation  $U$  on  $\mathcal{H}_B$  such that  $(I_{\mathcal{H}_A} \otimes U)|\psi\rangle = |\phi\rangle$ , where  $I_{\mathcal{H}_A}$  is the identity operator on  $\mathcal{H}_A$ .

We will also need another fundamental fact, shown by Uhlmann [8].

**Fact 12** (Uhlmann, [8]). Let  $\rho, \sigma \in \mathcal{H}_A$ . Let  $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$  be a purification of  $\rho$  and  $\dim(\mathcal{H}_A) \leq \dim(\mathcal{H}_B)$ . There exists a purification  $|\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$  of  $\sigma$  such that  $F(\rho, \sigma) = |\langle\phi|\psi\rangle|$ .

The approximate version of Schmidt decomposition that will be utilized in the present paper is as follows, which is called *approximate Schmidt rank*.

**Definition 11.** Let  $\epsilon > 0$ . Let  $|\psi\rangle$  be a pure state in  $\mathcal{H}_A \otimes \mathcal{H}_B$ . Define

$$\mathbf{S}\text{-rank}_\epsilon^{(A,B)}(|\psi\rangle) \stackrel{\text{def}}{=} \min\{\mathbf{S}\text{-rank}^{(A,B)}(|\phi\rangle) : |\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \text{ and } F(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|) \geq 1 - \epsilon\}.$$

For multipartite pure states, there are no Schmidt decompositions in general. But a weaker statement holds.

**Lemma 13.** Suppose  $|\psi\rangle$  is a pure state in  $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_k$ , and  $\rho_i$  is the reduced density matrix of  $|\psi\rangle$  in  $\mathcal{H}_i$ . Denote  $r_i = \mathbf{rank}(\rho_i)$ . If  $\{|\alpha_{ij}\rangle : j \in [r_i]\}$  are the eigenvectors of  $\rho_i$  corresponding to nonzero eigenvalues, then  $|\psi\rangle$  can be expressed as

$$|\psi\rangle = \sum_{j_1 \in [r_1], \dots, j_k \in [r_k]} a_{j_1 \dots j_k} |\alpha_{1j_1}\rangle \otimes \cdots \otimes |\alpha_{kj_k}\rangle,$$

where  $a_{j_1 \dots j_k}$ 's are complex coefficients.

*Proof.* For each  $i \in [k]$ , one can extend the vectors  $|\psi_{i1}\rangle, \dots, |\psi_{ir_i}\rangle$  to orthogonal basis  $|\psi_{i1}\rangle, \dots, |\psi_{iD_i}\rangle$  of  $\mathcal{H}_i$ , where  $D_i$  is the dimension of  $\mathcal{H}_i$ . One can then decompose  $|\psi\rangle$  according to the basis  $|\psi_{ij}\rangle : i \in [k], j \in [D_i]$ . The statement just says that  $|\psi\rangle$  does not have any component in  $|\psi_{ij}\rangle$ ,  $\forall i, \forall j > r_i$ . This is true because if  $|\psi\rangle$  has a nonzero component in  $|\psi_{ij}\rangle$  for some  $j > r_i$ , then when we compute the reduced density matrix of  $|\psi\rangle$  in  $\mathcal{H}_i$ , we get  $\rho_i$  with a positive component in  $|\psi_{ij}\rangle\langle\psi_{ij}|$ . Thus  $|\psi_{ij}\rangle$  is a eigenvector of  $\rho_i$  with a nonzero eigenvalue, contradictory to our assumption.  $\square$

### 3 Quantum Correlation Complexity of Multipartite States

In this section, we prove the results in Subsection 1.1 on quantum correlation complexity of multipartite states.

**Theorem 1 (Restated).** Suppose  $|\psi\rangle$  is a pure state in  $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ , and  $\rho_j$  is the reduced density matrices of  $|\psi\rangle$  in  $\mathcal{H}_j$ . Then

$$\text{QCorr}(|\psi\rangle) = \sum_{j=1}^k \lceil \log_2 \mathbf{rank}(\rho_j) \rceil.$$

*Proof.* Let  $r_j = \mathbf{rank}(\rho_j)$ . By Lemma 13, suppose that  $|\psi\rangle = \sum_{i_j \leq r_j} a_{i_1 \dots i_k} |\lambda_{i_1}\rangle \cdots |\lambda_{i_k}\rangle$ , where  $|\lambda_{ij}\rangle$  is the  $j$ -th eigenvector of  $\rho_j$ . Then the players can generate  $|\psi\rangle$  by local operations on the seed state  $|\psi'\rangle = \sum_{i_j \leq r_j} a_{i_1 \dots i_k} |i_1\rangle \cdots |i_k\rangle$ . Since this state takes  $\sum_{j=1}^k \lceil \log_2 \mathbf{rank}(\rho_j) \rceil$  number of qubits, we have shown that  $\text{QCorr}(|\psi\rangle) \leq \sum_{j=1}^k \lceil \log_2 \mathbf{rank}(\rho_j) \rceil$ .

For the other direction, let us assume the  $k$  players generate the target  $|\psi\rangle$  by local operations on an initial seed state  $\sigma$ , whose size is  $\text{QCorr}(|\psi\rangle)$ . First note that to generate a pure state, it is enough to have a pure state as the seed, since otherwise every pure state in the support of the mixed seed state can give the same target  $|\psi\rangle$ .

Now define the reduced density matrix of  $\sigma$  in the system  $A_j$  as  $\sigma_j$ , and assume that its rank is  $s_j$ . Then the size of  $\sigma$  is at least  $\sum_{j=1}^k \lceil \log_2 s_j \rceil$ , where the  $j$ -th summand bounds the number



of qubits for the  $j$ -th player's part of  $\sigma$ . Since local operations do not increase Schmidt rank, we know that  $s_j \geq r_j$ . Thus

$$\text{QCorr}(|\psi\rangle) \geq \sum_{j=1}^k \lceil \log_2 s_j \rceil \geq \sum_{j=1}^k \lceil \log_2 r_j \rceil = \sum_{j=1}^k \lceil \log_2 \text{rank}(\rho_j) \rceil.$$

□

As we mentioned earlier, generating a bipartite mixed quantum state  $\rho$  has the same cost as generating some purification of  $\rho$  [12]. In multipartite cases, however, this does not hold any more. The next theorem compares the quantum correlation complexity of generating a mixed state  $\rho$  and that of generating a purification.

**Theorem 2** (Restated). *Assume that  $\rho$  is a quantum state in  $\bigotimes_{i=1}^k \mathcal{H}_i$ . Then we have*

$$\text{QCorr}(\rho) \leq r(\rho) \leq \left(2 - \frac{2}{k}\right) \text{QCorr}(\rho),$$

where  $r(\rho)$  is the minimum  $\text{QCorr}(|\psi\rangle)$  over all purifications  $|\psi\rangle$  of  $\rho$ .

*Proof.* First, we have  $\text{QCorr}(\rho) \leq \text{QCorr}(|\psi\rangle)$  for any purification  $|\psi\rangle$  of  $\rho$ , thus  $\text{QCorr}(\rho) \leq r(\rho)$ .

Now for the other direction, suppose that a minimal seed state for generating  $\rho$  is  $\sigma$  with  $\text{size}(\sigma) = \text{QCorr}(\rho)$ . Let  $\sigma_i$  be the reduced density matrix of  $\sigma$  in  $\mathcal{H}_{A_i}$ , and suppose that  $n_i$  is the number of qubits of  $\sigma_i$ , so  $\text{QCorr}(\rho) = \sum_{i=1}^k n_i$ . Without loss of generality, assume that  $n_1 \leq \dots \leq n_k$ . Take any purification  $|\theta\rangle$  of  $\sigma$  in  $\mathcal{H}_{A_1} \otimes \dots \otimes \mathcal{H}_{A_{k-1}} \otimes \mathcal{H}_{A_k} \otimes \mathcal{H}_{A'_k}$ , where  $A'_k$  is the ancillary system introduced by  $A_k$ . In each player's part, the local operation can be assumed to be attaching some extra system, performing a unitary operation, and then tracing out part of system. Now if all players do not trace out any part of their systems, and act on initial state  $|\theta\rangle$  instead of  $\sigma$ , then the same protocol results in a pure state  $|\psi\rangle$ , which is a purification of  $\rho$ . In this way,  $\text{QCorr}(|\psi\rangle) \leq \text{QCorr}(|\theta\rangle)$ .

According to Theorem 1, we have  $\text{QCorr}(|\theta\rangle) = \sum_{i=1}^k \lceil \log_2 r_i \rceil$ , where  $r_i$  is the dimension of  $\sigma_i$  for  $i \leq k-1$ , and  $r_k$  is the dimension of  $\text{tr}_{\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_{k-1}} |\theta\rangle\langle\theta|$ . Note that

$$r_i \leq 2^{n_i}, \quad \forall i \leq k-1, \quad \text{and} \quad r_k \leq 2^{n_1 + \dots + n_{k-1}},$$

where the last inequality uses the fact that  $|\theta\rangle$  is a pure state. Thus, it follows that

$$\text{QCorr}(|\psi\rangle) \leq \text{QCorr}(|\theta\rangle) = \sum_{i=1}^k \lceil \log_2 r_i \rceil \leq 2 \sum_{i=1}^{k-1} n_i \leq \left(2 - \frac{2}{k}\right) \sum_{i=1}^k n_i = \left(2 - \frac{2}{k}\right) \text{QCorr}(\rho).$$

□

In the above theorem, the left inequality is tight when  $\rho$  is a pure state. The following proposition shows that the right inequality is also tight by giving an example of tripartite state  $\rho$  with  $\text{QCorr}(\rho) = 3$  and  $r(\rho) = 4$ . Recall that the 3-qubit GHZ state is  $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$  and the 3-qubit W state is  $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ .

**Proposition 14.** *For  $\rho_0 = \frac{1}{2}|GHZ\rangle\langle GHZ| + \frac{1}{2}|W\rangle\langle W|$ , we have  $\text{QCorr}(\rho_0) = 3$  and  $r(\rho_0) = 4$ .*

*Proof.* Since  $\rho_0$  is a 3-qubit state, the three players can simply share itself as the seed (and then do nothing), so  $\text{QCorr}(\rho_0) \leq 3$ . We will next show that  $r(\rho_0) = 4$ , which implies  $\text{QCorr}(\rho_0) \geq 3$  by Theorem 2. Therefore  $\text{QCorr}(\rho_0) = 3$ .

We now prove that  $r(\rho_0) = 4$ . Suppose the three qubits of  $\rho_0$  are possessed by Alice, Bob, and Charlie respectively. One simple purification is

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|GHZ\rangle|1\rangle + \frac{1}{\sqrt{2}}|W\rangle|0\rangle,$$

where the last qubit is introduced by one player, say, Charlie. Since  $|\psi_0\rangle$  has only 4 qubits,  $r(\rho_0) \leq 4$ . We shall prove that  $r(\rho_0) \geq 4$ .

Suppose the three qubits of  $\rho_0$  are possessed by Alice, Bob, and Charlie respectively. For convenience, we call these three qubits the main system. Then an arbitrary purification of  $\rho_0$  in  $\mathcal{H}_A \otimes \mathcal{H}_{A_1} \otimes \mathcal{H}_B \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_C \otimes \mathcal{H}_{C_1}$  could be expressed as

$$|\psi\rangle = \frac{1}{\sqrt{2}}|GHZ\rangle|u_0\rangle + \frac{1}{\sqrt{2}}|W\rangle|u_1\rangle,$$

where  $|u_0\rangle$  and  $|u_1\rangle$  are orthogonal, and they are composed by all the ancillary systems introduced by the three players. Note that it is possible that some of the players do not have ancillary systems. Without loss of generality, we suppose some of the qubits in  $|u_i\rangle$  belong to Alice. We trace out the two qubits of Bob and Charlie in the main systems from  $|\psi\rangle$ , and get

$$\rho_a = \text{tr}_{\mathcal{H}_B \otimes \mathcal{H}_C} |\psi\rangle\langle\psi| \quad (5)$$

$$= \left( \frac{1}{2}|0\rangle|u_0\rangle + \frac{1}{\sqrt{6}}|1\rangle|u_1\rangle \right) \left( \frac{1}{2}\langle 0|\langle u_0| + \frac{1}{\sqrt{6}}\langle 1|\langle u_1| \right) \quad (6)$$

$$+ \frac{1}{4}|1\rangle\langle 1| \otimes |u_0\rangle\langle u_0| + \frac{1}{3}|0\rangle\langle 0| \otimes |u_1\rangle\langle u_1|, \quad (7)$$

where the first qubit belongs to Alice, and the rest is all the ancillary systems combined. Continue to trace out Bob's ancillary system and Charlie's ancillary system, then we obtain Alice's reduced density matrix  $\rho'_a$ . Similarly, we can define  $\rho'_b$  or  $\rho'_c$ , provided Bob or Charlie has a nontrivial part in  $|u_i\rangle$ .

We now prove that at least one of  $\rho'_a$ ,  $\rho'_b$  and  $\rho'_c$  has a rank at least 3. If this is the case, say  $\text{rank}(\rho'_a) \geq 3$ , then Alice needs at least 2 qubits. Since Bob and Charlie each needs at least 1 qubit,  $\text{QCorr}(|\psi\rangle) \geq 4$ .

If  $|u_i\rangle$  is only at Alice's side, *i.e.*, only Alice introduces an ancillary system, then  $\rho'_a = \rho_a$ , which has rank 3. Now suppose that Bob also introduces an ancillary system. We claim that if one of  $|u_0\rangle$  and  $|u_1\rangle$  is not a product state across  $(A, BC)$ , then one of  $\rho'_a$ ,  $\rho'_b$  and  $\rho'_c$  has rank at least 3. Indeed, suppose  $|u_0\rangle$  is not a product state across  $(A, BC)$ , then  $\text{rank}(\text{tr}_{\mathcal{H}_{B_1} \otimes \mathcal{H}_{C_1}} |u_0\rangle\langle u_0|) \geq 2$ . Note that the three components in (5) are orthogonal, thus  $\text{rank}(\rho'_a) \geq \text{rank}(\text{tr}_{\mathcal{H}_{B_1} \otimes \mathcal{H}_{C_1}} |u_0\rangle\langle u_0|) + \text{rank}(\text{tr}_{\mathcal{H}_{B_1} \otimes \mathcal{H}_{C_1}} |u_1\rangle\langle u_1|)$ , which means  $\text{rank}(\rho'_a) \geq 3$ . Therefore, we only need to take care of the situation where  $|u_0\rangle$  and  $|u_1\rangle$  are product states. Since they are orthogonal, without loss of generality we could express them as  $|u_0\rangle = |u_{0,a}\rangle|v_{0,bc}\rangle$  and  $|u_1\rangle = |u_{1,a}\rangle|v_{1,bc}\rangle$ , where  $|u_{0,a}\rangle, |u_{1,a}\rangle \in \mathcal{H}_{A_1}$ ,  $|v_{0,bc}\rangle, |v_{1,bc}\rangle \in \mathcal{H}_{B_1} \otimes \mathcal{H}_{C_1}$ , with either  $\langle u_{0,a}|u_{1,a}\rangle = 0$  or  $\langle v_{0,bc}|v_{1,bc}\rangle = 0$ . In this way,

$$|\psi\rangle = \frac{1}{2}(|000\rangle + |111\rangle)|u_{0,a}\rangle|v_{0,bc}\rangle + \frac{1}{\sqrt{6}}(|001\rangle + |010\rangle + |001\rangle)|u_{1,a}\rangle|v_{1,bc}\rangle.$$

It is not difficult to verify that the rank of  $\rho'_{bc} = \text{tr}_{\mathcal{H}_A \otimes \mathcal{H}_{A_1}} |\psi\rangle\langle\psi|$  is at least 3. Meanwhile, it holds that  $\text{rank}(\rho'_{bc}) = \text{rank}(\rho'_a)$ . Hence,  $\text{rank}(\rho'_a) \geq 3$ , and this completes the proof.  $\square$

Next we consider the other extreme, when  $\rho$  is a multipartite classical state, *i.e.*, a multipartite probability distribution. Recall that for a classical distribution  $P$  on  $\mathcal{X}$ , we often identify it with  $\rho = \sum_x P(x) |x\rangle\langle x|$ . Also recall that for a nonnegative tensor  $P = [P(x_1, \dots, x_k)]_{x_1, \dots, x_k}$ , its PSD-rank  $\text{rank}_{\text{psd}}^{(k)}(P)$  is the minimum  $r$  *s.t.* there are  $r \times r$  PSD matrices  $C_{x_1}^{(1)}, \dots, C_{x_k}^{(k)} \succeq 0$  with  $P(x_1, \dots, x_k) = \sum_{i,j=1}^r C_{x_1}^{(1)}(i,j) \cdots C_{x_k}^{(k)}(i,j)$ .

**Theorem 3 (Restated).** *Suppose  $P = [P(x_1, \dots, x_k)]_{x_1, \dots, x_k}$  is a probability distribution on  $\mathcal{X}_1 \times \dots \times \mathcal{X}_k$ . Then we have*

$$\frac{k}{2k-2} \lceil \log_2 \text{rank}_{\text{psd}}^{(k)}(P) \rceil \leq \text{QCorr}(P) \leq k \lceil \log_2 \text{rank}_{\text{psd}}^{(k)}(P) \rceil.$$

*Proof.* We first prove the right inequality. Let  $r = \text{rank}_{\text{psd}}^{(k)}(P)$ , then there exist positive semi-definite matrices  $\{C_{x_i}^{(i)} : i \in [k], x_i \in \mathcal{X}_i\}$  *s.t.* for any  $x = (x_1, \dots, x_k)$ , it holds that  $P(x) = \sum_{i,j=1}^r \prod_{t=1}^k C_{x_t}^{(t)}(i,j)$ . For  $i \in [r]$ , let  $|u_{x_t}^i\rangle$  be the  $i$ -th column of  $\sqrt{C_{x_t}^{(t)}}$ . Then we have that  $\langle u_{x_t}^j | u_{x_t}^i \rangle = C_{x_t}^{(t)}(i,j)$ . We now define a pure state  $|\psi\rangle \in \bigotimes_{t=1}^r (\mathcal{H}_{A_t} \otimes \mathcal{H}_{A'_t} \otimes \mathcal{H}_{A''_t})$  as follows.

$$|\psi\rangle = \sum_{i=1}^r \bigotimes_{t=1}^k \sum_{x_t} (|x_t\rangle \otimes |x_t\rangle \otimes |u_{x_t}^i\rangle).$$

For each  $t$ , tracing out the second and the third registers gives

$$\begin{aligned} & \text{tr}_{\mathcal{H}_{A'_t} \otimes \mathcal{H}_{A''_t}} |\psi\rangle\langle\psi| \\ &= \sum_{x_1, \dots, x_k} |x_1\rangle\langle x_1| \otimes \cdots \otimes |x_k\rangle\langle x_k| \left( \sum_{i,j=1}^r \prod_{t=1}^k \langle u_{x_t}^j | u_{x_t}^i \rangle \right) \\ &= \sum_{x_1, \dots, x_k} |x_1\rangle\langle x_1| \otimes \cdots \otimes |x_k\rangle\langle x_k| \left( \sum_{i,j=1}^r \prod_{t=1}^k C_{x_t}^{(t)}(i,j) \right) \\ &= \sum_{x_1, \dots, x_k} P(x_1, \dots, x_k) \cdot |x_1\rangle\langle x_1| \otimes \cdots \otimes |x_k\rangle\langle x_k|. \end{aligned}$$

Thus  $|\psi\rangle$  is actually a purification of  $\rho$ , and Theorem 2 implies that  $\text{QCorr}(\rho) \leq \text{QCorr}(|\psi\rangle)$ . Further note that  $\text{QCorr}(|\psi\rangle) \leq k \lceil \log_2 r \rceil$  by Theorem 1. We thus show that  $\text{QCorr}(\rho) \leq k \lceil \log_2 r \rceil$ .

For the left inequality, suppose  $|\psi'\rangle$  is a pure state in  $\bigotimes_{i=1}^k (\mathcal{H}_{A_i} \otimes \mathcal{H}_{A'_i})$  that achieves the optimum of  $r(\rho)$  in Theorem 2, then this theorem tells us that

$$\text{QCorr}(\rho) \geq \frac{k}{2k-2} \text{QCorr}(|\psi'\rangle) = \frac{k}{2k-2} \sum_{i=1}^k \lceil \log_2 r_i \rceil \geq \frac{k \log_2 (\prod_{i=1}^k r_i)}{2k-2}, \quad (8)$$

where  $r_i$  is the dimension of the reduced density matrix of  $|\psi'\rangle$  on the  $i$ -th player. According to Lemma 13,  $|\psi'\rangle$  could be expressed as

$$|\psi'\rangle = \sum_{i=1}^R a_i |\alpha_i^1\rangle \cdots |\alpha_i^k\rangle.$$

Here  $R = \prod_{j=1}^k r_j$ , and for  $i \in [R]$ ,  $|\alpha_i^j\rangle \in \mathcal{H}_{A_j} \otimes \mathcal{H}_{A'_j}$ . It should be pointed out that for different  $i$  and  $i'$ ,  $|\alpha_i^j\rangle$  and  $|\alpha_{i'}^j\rangle$  might be the same. In this way,  $|\psi'\rangle$  could also be written as

$$|\psi'\rangle = \sum_{i=1}^R \bigotimes_{j=1}^k \left( \sum_{x_j} |x_j\rangle \otimes |u_{x_j}^i\rangle \right).$$

Recall that  $|\psi'\rangle$  is a purification of  $\rho$ , so

$$\begin{aligned} \rho &= \text{tr}_{\mathcal{H}_{A'_1} \otimes \dots \otimes \mathcal{H}_{A'_k}} |\psi'\rangle \langle \psi'| \\ &= \sum_{x_1, \dots, x_k} |x_1\rangle \langle x_1| \otimes \cdots \otimes |x_k\rangle \langle x_k| \left( \sum_{i, i'=1}^R \prod_{j=1}^k \langle u_{x_j}^{i'} | u_{x_j}^i \rangle \right) \\ &= \sum_{x_1, \dots, x_k} P(x_1, \dots, x_k) |x_1\rangle \langle x_1| \otimes \cdots \otimes |x_k\rangle \langle x_k|. \end{aligned}$$

Note that for any  $x$ , the  $R \times R$  matrix  $C_x$  with  $C_x(j, i) = \langle u_x^j | u_x^i \rangle$  is positive. So by the definition of PSD-rank, we have that  $\text{rank}_{\text{psd}}^{(k)}(P) \leq R = \prod_{j=1}^k r_j$ . Combining this result with Eq.(8), we get that  $\text{QCorr}(\rho) \geq \frac{k}{2k-2} \lceil \log_2 \text{rank}_{\text{psd}}^{(k)}(P) \rceil$ , which completes the proof.  $\square$

## 4 Quantum Communication Complexity of Multipartite States

In this section, we study communication complexity of generating multipartite states and prove the results in Section 1.2.

**Theorem 4** (Restated). *Suppose  $|\psi\rangle$  is a  $k$ -partite pure state, and  $M(|\psi\rangle) = \sum_{j=1}^k \lceil \log_2 \text{rank}(\rho_j) \rceil$  where  $\rho_i$  is  $|\psi\rangle$  reduced to Player  $i$ 's space. Then*

$$\frac{1}{2} M(|\psi\rangle) \leq \text{QComm}(|\psi\rangle) \leq \frac{k-1}{k} M(|\psi\rangle).$$

*Proof.* Let us prove the upper bound first. By Theorem 1, we can assume that the players can generate  $\rho$  by local operations on the seed state  $\sigma$  of size  $M(|\psi\rangle)$ . Suppose that Player  $i$ 's part of  $\sigma$  has the largest number of qubits, then this player can prepare  $\sigma$  and send to other players their parts. The communication cost is thus at most  $\frac{k-1}{k} M(|\psi\rangle)$ .

For the lower bound, suppose that Player  $i$  and Player  $j$  communicate  $c_{ij}$  qubits in an optimal communication protocol generating  $|\psi\rangle$ , starting from a product state. Considering the linearity of quantum operations and that the target state is pure, we can assume without loss of generality that the seed state is also pure. Denote by  $r_i = \text{rank}(\rho_i)$  where  $\rho_i$  is  $|\psi\rangle$  reduced to Player  $i$ 's space.

Since exchanging  $r$  qubits can only increase the Schmidt rank between Player  $i$  and the rest of the players by at most  $2^r$ , we have that

$$r_i \leq 2^{\sum_{j:j \neq i} c_{ij}}.$$

Putting communication among all pairs of players together, we have

$$\text{QComm}(|\psi\rangle) = \sum_{\{i,j\}:i \neq j} c_{ij} = \frac{1}{2} \sum_i \sum_{j:j \neq i} c_{ij} \geq \frac{1}{2} \sum_i \lceil \log_2 r_i \rceil \geq \frac{1}{2} \text{M}(|\psi\rangle).$$

□

Both bounds in the above theorem are tight. For the upper bound, consider the 3-qubit GHZ state  $|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$  shared by Alice, Bob and Charlie. It is not hard to see that  $\text{M}(|\psi\rangle) = 3$  and  $\text{QComm}(|\psi\rangle) = 2$ . For the lower bound, consider an EPR pair  $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$  shared by two players. It has  $\text{M}(|\psi\rangle) = 2$  and  $\text{QComm}(|\psi\rangle) = 1$ .

In Theorem 2 and its later comment on tightness of the bounds we have seen that the correlation complexity of a mixed quantum state  $\rho$  is in general different than that of (even a best) purification of  $\rho$ . The next theorem shows that for communication complexity, generating a mixed quantum state is the same as generating a purification of it.

**Theorem 5** (Restated). *For any  $k$ -partite quantum state  $\rho$ ,*

$$\text{QComm}(\rho) = \min\{\text{QComm}(|\psi\rangle) : |\psi\rangle \text{ is a purification of } \rho\}.$$

*Proof.* It is clear that for any purification  $|\psi\rangle$ ,  $\text{QComm}(\rho) \leq \text{QComm}(|\psi\rangle)$  since one can just generate  $|\psi\rangle$  and then trace out some part to get  $\rho$ .

For the other direction, suppose  $r = \text{QComm}(\rho)$ , then starting from  $\otimes_{i=1}^k |0\rangle$ , the players can generate  $\rho$  by local operations and communicating  $r$  qubits. Here all local operations can be assumed to be first to append some ancilla and then perform a unitary operation and finally trace out some parts. If the players do not trace out any part, then at the end of the protocol, they would have a pure state as a purification  $|\psi\rangle$  of  $\rho$ . Thus  $\text{QComm}(\rho) \geq \text{QComm}(|\psi\rangle)$ . □

The following result compares  $\text{QCorr}(\rho)$  and  $\text{QComm}(\rho)$  for general multipartite quantum states.

**Corollary 6** (Restated). *For any  $k$ -partite quantum state  $\rho$ , it holds that*

$$\frac{k}{k-1} \text{QComm}(\rho) \leq \text{QCorr}(\rho) \leq 2 \text{QComm}(\rho).$$

*Proof.* The left inequality can be easily proved using the same argument as the lower bound proof of Theorem 4.

For the right inequality, according to Theorem 5, we could find a purification  $|\psi\rangle$  of  $\rho$  in  $\bigotimes_{i=1}^k (\mathcal{H}_{A_i} \otimes \mathcal{H}_{A'_i})$  such that  $\text{QComm}(\rho) = \text{QComm}(|\psi\rangle)$ . Then Theorem 4 indicates that  $\text{QComm}(|\psi\rangle) \geq \frac{1}{2} \text{QCorr}(|\psi\rangle)$ . Combing these results with Theorem 2, we obtain that

$$\text{QCorr}(\rho) \leq \text{QCorr}(|\psi\rangle) \leq 2 \text{QComm}(|\psi\rangle) = 2 \text{QComm}(\rho).$$

□

## 5 Approximate Quantum Correlation Complexity of Bipartite States

In this section, we study the correlation complexity of generating bipartite states approximately, and prove the results mentioned in Section 1.3.1. We will first consider two extreme cases: quantum pure states and classical distributions, and then general quantum mixed states.

### Quantum pure states.

For quantum pure states, we will first show that the following two approximations are equivalent. Recall that for a state  $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$ ,

$$\text{QCorr}_\epsilon(\rho) = \min\{\text{QCorr}(\rho') : \rho' \in \mathcal{H}_{AB}, F(\rho, \rho') \geq 1 - \epsilon\},$$

and

$$\begin{aligned} \text{QCorr}'_\epsilon(\rho) &= \min\{\text{QCorr}_\epsilon^{\text{pure}}(|\varphi\rangle) : |\varphi\rangle \in \mathcal{H}_{A_1ABB_1}, \rho = \text{tr}_{\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}} |\varphi\rangle\langle\varphi|\} \\ &= \min\{\lceil \log_2 \mathbf{S}\text{-rank}_\epsilon(|\varphi\rangle) \rceil : |\varphi\rangle \in \mathcal{H}_{A_1ABB_1}, \rho = \text{tr}_{\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}} |\varphi\rangle\langle\varphi|\}. \end{aligned}$$

We will need a result in [6] which says that pure states can be optimally approximated by given other *pure* state.

**Lemma 15** ([6]). *For a bipartite pure state  $|\psi\rangle$  with Schmidt coefficients  $\lambda_1 \geq \dots \geq \lambda_N$ ,  $\text{QCorr}_\epsilon(|\psi\rangle) = \text{QCorr}_\epsilon^{\text{pure}}(|\psi\rangle) = \lceil \log_2 r \rceil$ , where  $r$  is the minimum integer s.t.  $\sum_{i=1}^r \lambda_i^2 \geq (1 - \epsilon)^2$ .*

**Proposition 16.** *For any quantum state  $\rho$  in  $\mathcal{H}_A \otimes \mathcal{H}_B$ ,  $\text{QCorr}_\epsilon(\rho) = \text{QCorr}'_\epsilon(\rho)$ .*

*Proof.*  $\text{QCorr}_\epsilon(\rho) \geq \text{QCorr}'_\epsilon(\rho)$ : Suppose that  $\rho' \in \mathcal{H}_A \otimes \mathcal{H}_B$ ,  $F(\rho, \rho') \geq 1 - \epsilon$  and  $\text{QCorr}_\epsilon(\rho) = \text{QCorr}(\rho')$ . By Lemma 2.2 of [6], there is a purification  $|\psi\rangle$  in  $A_1ABB_1$  of  $\rho'$  s.t.  $\text{QCorr}(\rho') = \lceil \log_2 \mathbf{S}\text{-rank}(|\psi\rangle) \rceil$ . By Uhlmann's theorem, there exists a purification of  $\rho$  in  $A_1ABB_1$ , say  $|\alpha\rangle$ , and  $F(|\alpha\rangle\langle\alpha|, |\psi\rangle\langle\psi|) = F(\rho, \rho') \geq 1 - \epsilon$ . (We assume that the  $|\alpha\rangle$  and  $|\psi\rangle$  are in the same extended space  $\mathcal{H}_{A_1ABB_1}$  since otherwise we can use the union of the two extended spaces.) Thus

$$\text{QCorr}'_\epsilon(\rho) \leq \lceil \log_2 \mathbf{S}\text{-rank}_\epsilon(|\alpha\rangle) \rceil \leq \lceil \log_2 \mathbf{S}\text{-rank}(|\psi\rangle) \rceil = \text{QCorr}_\epsilon(\rho).$$

$\text{QCorr}_\epsilon(\rho) \leq \text{QCorr}'_\epsilon(\rho)$ : Suppose  $\text{QCorr}'_\epsilon(\rho) = \lceil \log_2 \mathbf{S}\text{-rank}_\epsilon(|\varphi\rangle) \rceil$ , and  $\rho = \text{tr}_{\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}} |\varphi\rangle\langle\varphi|$ . Then one can find another pure state  $|\beta\rangle$  in  $A_1ABB_1$ , such that  $\lceil \log_2 \mathbf{S}\text{-rank}(|\beta\rangle) \rceil = \lceil \log_2 \mathbf{S}\text{-rank}_\epsilon(|\varphi\rangle) \rceil = \text{QCorr}'_\epsilon(\rho)$ , and  $F(|\beta\rangle\langle\beta|, |\varphi\rangle\langle\varphi|) \geq 1 - \epsilon$ . Since partial trace does not decrease the fidelity [8], we know that  $F(\text{tr}_{\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}} |\beta\rangle\langle\beta|, \rho) \geq 1 - \epsilon$ . By the definition of  $\text{QCorr}_\epsilon(\rho)$ , it holds that  $\text{QCorr}'_\epsilon(\rho) \geq \text{QCorr}_\epsilon(\rho)$ , which completes the proof.  $\square$

### Classical distributions.

Next we consider to approximate classical distributions. Recall that Lemma 15 implies that the most efficient approximate generation of a pure state can be achieved by another pure state. In the same spirit, the following theorem shows that the most efficient approximate generation of a classical state can be achieved by another classical state, and the correlation complexity is completely determined by the approximate PSD-rank.

**Theorem 7** (Restated). *For any classical state  $P = [P(x, y)]_{x, y}$ ,*

$$\text{QCorr}_\epsilon(P) = \text{QCorr}_\epsilon^{\text{cla}}(P) = \lceil \log_2 \text{rank}_{\text{psd}, \epsilon}(P) \rceil.$$

*Proof.* For the first equality, we only need to prove that  $\text{QCorr}_\epsilon(P) \geq \text{QCorr}_\epsilon^{\text{cla}}(P)$  (since the other direction holds by definition). Given an approximation  $\rho'$  to  $P$  with  $F(P, \rho') \geq 1 - \epsilon$  and  $\text{size}(\rho') = \text{QCorr}_\epsilon(P)$ , we measure  $\rho'$  in the computational basis of  $\mathcal{H}_A \otimes \mathcal{H}_B$  and get a probability distribution  $P'$ . Note that the same measurement does not change  $P$ . Since no operation can decrease the fidelity of two states, we have  $F(P, P') \geq F(P, \rho') \geq 1 - \epsilon$ .

The second equality is immediate from their definitions.  $\square$

## General quantum mixed states.

We now turn to the case of general bipartite  $\sigma$ . By combining Theorem 1.2 of [6] and Proposition 16, we have the following characterization of  $\text{QCorr}_\epsilon(\sigma)$ .

**Theorem 8** (Restated). *Let  $\sigma$  be an arbitrary quantum state in  $\mathcal{H}_A \otimes \mathcal{H}_B$ , and  $0 < \epsilon < 1$ . Then  $\text{QCorr}_\epsilon(\sigma) = \lceil \log_2 r \rceil$ , where  $r$  is the minimum integer s.t. there exist a collection of matrices,  $\{A_x\}$ 's and  $\{B_y\}$ 's of the same column number  $l \geq r$ , satisfying the following conditions.*

1. *The matrices relate to  $\sigma$  by the following equation.*

$$\sigma = \sum_{x, x'; y, y'} |x\rangle\langle x'| \otimes |y\rangle\langle y'| \cdot \text{tr}\left((A_{x'}^\dagger A_x)^T (B_{y'}^\dagger B_y)\right). \quad (9)$$

2. *Denoting the  $i$ -th column of any matrix  $M$  by  $|M(i)\rangle$ , then*

$$\sum_x \langle A_x(i) | A_x(j) \rangle = \sum_y \langle B_y(i) | B_y(j) \rangle = 0, \quad (10)$$

- 3.

$$\sum_{i=1}^r \left( \sum_x \langle A_x(i) | A_x(i) \rangle \right) \left( \sum_y \langle B_y(i) | B_y(i) \rangle \right) \geq 1 - \epsilon, \quad (11)$$

*Proof.* Suppose  $\text{QCorr}_\epsilon(\sigma) = \lceil \log_2 \text{S-rank}_\epsilon(|\psi\rangle) \rceil$  where  $|\psi\rangle$  is a purification of  $\sigma$ , given by Proposition 16. Put  $t = \text{S-rank}_\epsilon(|\psi\rangle)$ . Suppose the Schmidt decomposition of  $|\psi\rangle$  is

$$|\psi\rangle = \sum_{i=1}^s \left( \sum_x |x\rangle \otimes |v_x^i\rangle \right) \otimes \left( \sum_y |y\rangle \otimes |w_y^i\rangle \right), \quad (12)$$

thus the Schmidt coefficients are  $a_i = \sum_x \langle v_x^i | v_x^i \rangle \sum_y \langle w_y^i | w_y^i \rangle$ ,  $1 \leq i \leq s$ . For each  $x$ , set matrices  $A_x \stackrel{\text{def}}{=} (|v_x^1\rangle, |v_x^2\rangle, \dots, |v_x^s\rangle)$ . Similarly, for each  $y$  set matrices  $B_y \stackrel{\text{def}}{=} (|w_y^1\rangle, |w_y^2\rangle, \dots, |w_y^s\rangle)$ . Then it can be verified that Eq.(9) holds. In addition, the orthogonality of  $\sum_x |x\rangle \otimes |v_x^i\rangle$  (and that of  $\sum_y |y\rangle \otimes |w_y^i\rangle$ ) for different  $i$ 's translates to Eq.(10).

Without loss of generality, we assume that the coefficients  $a_i$ 's are in the decreasing order. By Lemma 5.1 of [6], we have that

$$\sum_{i=1}^t \left( \sum_x \langle A_x(i) | A_x(i) \rangle \right) \left( \sum_y \langle B_y(i) | B_y(i) \rangle \right) = \sum_{i=1}^t a_i \geq 1 - \epsilon.$$

Therefore all three conditions hold for  $\{A_x\}$  and  $\{B_y\}$ , implying that  $r \leq t$ , and that  $\text{QCorr}_\epsilon(\sigma) \geq \lceil \log_2 r \rceil$ .

For the other direction, given that matrices  $\{A_x\}$ 's and  $\{B_y\}$ 's satisfy the requirements, it can be verified that

$$|\tilde{\psi}\rangle = \sum_{i=1}^l \left( \sum_x |x\rangle \otimes |A_x(i)\rangle \right) \otimes \left( \sum_y |y\rangle \otimes |B_y(i)\rangle \right) \quad (13)$$

is a purification of  $\sigma$  in  $\mathcal{H}_{A_1} \otimes \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_{B_1}$ . Again assuming the decreasing order of the Schmidt coefficients and taking the  $r$  leading terms,

$$|\tilde{\psi}'\rangle = \sum_{i=1}^r \left( \sum_x |x\rangle \otimes |A_x(i)\rangle \right) \otimes \left( \sum_y |y\rangle \otimes |B_y(i)\rangle \right), \quad (14)$$

then Eq.(11) means that  $|\langle \tilde{\psi} | \tilde{\psi}' \rangle| \geq 1 - \epsilon$ . Since  $\mathbf{S}\text{-rank}(|\tilde{\psi}'\rangle) \leq r$ , it holds that  $\mathbf{S}\text{-rank}_\epsilon(|\tilde{\psi}\rangle) \leq r$ , and according to Proposition 16 we know that  $\text{QCorr}_\epsilon(\sigma) \leq \lceil \log_2 r \rceil$ , which completes the proof.  $\square$

## 6 Approximate Quantum Correlation Complexity of Multipartite Pure States

In this section, we consider approximation in generating multipartite pure states, and prove the results in Section 1.3.2. Recall that for a  $k$ -partite pure state  $|\psi\rangle$ ,  $r_i^{(\epsilon)} = \mathbf{S}\text{-rank}_\epsilon^{(A_i, A_{-i})}(|\psi\rangle)$ .

**Theorem 9** (Restated). *Let  $|\psi\rangle \in \bigotimes_{i=1}^k \mathcal{H}_i$  be a  $k$ -partite state,  $\epsilon > 0$ , and  $M_\epsilon(|\psi\rangle) = \sum_{i=1}^k \lceil \log_2 r_i^{(\epsilon)} \rceil$ . Then*

$$M_\epsilon(|\psi\rangle) \leq \text{QCorr}_\epsilon^{\text{pure}}(|\psi\rangle) \leq M_{\epsilon/k}(|\psi\rangle).$$

*Proof.* Lower bound: Suppose  $|\phi\rangle$  is a pure state in  $\bigotimes_{i=1}^k \mathcal{H}_{A_i}$  such that  $\text{QCorr}_\epsilon^{\text{pure}}(|\psi\rangle) = \text{QCorr}(|\phi\rangle)$  and  $F(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|) \geq 1 - \epsilon$ . Suppose  $\sigma_i$  is the reduced density matrix of  $|\phi\rangle$  in Player  $i$ 's system, and its rank is  $s_i$ , which is also  $\mathbf{S}\text{-rank}_\epsilon^{(A_i, A_{-i})}(|\psi\rangle)$ . Then it holds that  $s_i \geq r_i^{(\epsilon)}$ . According to Theorem 1, we have that

$$\text{QCorr}_\epsilon^{\text{pure}}(|\psi\rangle) = \text{QCorr}(|\phi\rangle) = \sum_{i=1}^K \lceil \log_2 s_i \rceil \geq M_\epsilon(|\psi\rangle).$$

Upper bound: Lemma 13 shows that  $|\psi\rangle$  could be written as

$$|\psi\rangle = \sum_{i_j \leq r_j} a_{i_1 \dots i_k} |\lambda_{i_1}\rangle \cdots |\lambda_{i_k}\rangle$$



where  $|\lambda_{i_j}\rangle$  is the  $i_j$ -th eigenvector of  $\rho_j$ . Arrange  $i_j$  in decreasing order of the eigenvalues of  $\rho_j$ . Since  $|\psi\rangle$  is a pure state,

$$\sum_{i_1, \dots, i_k} |a_{i_1 \dots i_k}|^2 = 1.$$

According to the definition of  $r_j^{(\epsilon)}$ , we have that

$$\sum_{i_j=r_j^{(\epsilon/k)}+1}^{r_j} \sum_{i-j} |a_{i_1 \dots i_k}|^2 \leq \epsilon/k,$$

where we have used Lemma 5.1 of [6] and the fact that

$$\langle \lambda_{i_j} | \rho_j | \lambda_{i_j} \rangle = \sum_{i-j} |a_{i_1 \dots i_k}|^2.$$

Thus,

$$\sum_{i_1=1}^{r_1^{(\epsilon/k)}} \cdots \sum_{i_k=1}^{r_k^{(\epsilon/k)}} |a_{i_1 \dots i_k}|^2 \geq 1 - \sum_{j=1}^k \sum_{i_j=r_j^{(\epsilon/k)}+1}^{r_j} \sum_{i-j} |a_{i_1 \dots i_k}|^2 \geq 1 - \epsilon.$$

We now consider a pure state defined as

$$|\phi'\rangle = \frac{1}{\sqrt{m}} \sum_{i_1=1}^{r_1^{(\epsilon/k)}} \cdots \sum_{i_k=1}^{r_k^{(\epsilon/k)}} a_{i_1 \dots i_k} |\lambda_{i_1}\rangle \cdots |\lambda_{i_k}\rangle,$$

where  $m = \sum_{i_1=1}^{r_1^{(\epsilon/k)}} \cdots \sum_{i_k=1}^{r_k^{(\epsilon/k)}} |a_{i_1 \dots i_k}|^2$ . It is not difficult to prove that  $F(|\psi\rangle\langle\psi|, |\phi'\rangle\langle\phi'|) \geq \sqrt{1-2\epsilon} \approx 1 - \epsilon$ . Moreover, according to Theorem 1, it holds that  $\text{QCorr}(|\phi'\rangle) \leq M_{\epsilon/k}(|\psi\rangle)$ . According to the definition of  $\text{QCorr}_\epsilon^{\text{pure}}(|\psi\rangle)$ , we obtain that  $\text{QCorr}_\epsilon^{\text{pure}}(|\psi\rangle) \leq M_{\epsilon/k}(|\psi\rangle)$ .  $\square$

From the proof we can see that the upper bound can be generalized to the following.

**Theorem 17.** *Suppose*

$$R = \min_{S_1, \dots, S_k} \left\{ \prod_{i=1}^k |S_i| : S_i \subset [r_i], \sum_{i_1 \in S_1, \dots, i_k \in S_k} |a_{i_1 \dots i_k}|^2 \geq 1 - \epsilon \right\}.$$

*Then  $\text{QCorr}_\epsilon^{\text{pure}}(|\psi\rangle) \leq \log_2 \lceil R \rceil$ .*

Finally, we consider the relationship between  $\text{QCorr}_\epsilon(|\psi\rangle)$  and  $\text{QCorr}_\epsilon^{\text{pure}}(|\psi\rangle)$ .

**Theorem 10 (Restated).** *Let  $|\psi\rangle \in \mathcal{H}_{A_1} \otimes \cdots \otimes \mathcal{H}_{A_k}$  be a pure state and  $\epsilon > 0$ . Then*

$$\frac{k}{2k-2} \text{QCorr}_{k\epsilon}^{\text{pure}}(|\psi\rangle) \leq \text{QCorr}_\epsilon(|\psi\rangle) \leq \text{QCorr}_\epsilon^{\text{pure}}(|\psi\rangle).$$

*Proof.* The second inequality  $\text{QCorr}_\epsilon(|\psi\rangle) \leq \text{QCorr}_\epsilon^{\text{pure}}(|\psi\rangle)$  holds by definition. For the first inequality, according to the definition of  $\text{QCorr}_\epsilon(|\psi\rangle)$ , there exists a  $\rho \in \mathcal{H}_{A_1} \otimes \cdots \otimes \mathcal{H}_{A_k}$  such that  $\text{QCorr}_\epsilon(|\psi\rangle) = \text{QCorr}(\rho)$  and  $F(|\psi\rangle\langle\psi|, \rho) \geq 1 - \epsilon$ . Then Theorem 2 implies that there exists a purification  $|\phi\rangle \in \bigotimes_{i=1}^k (\mathcal{H}_{A_i} \otimes \mathcal{H}_{A'_i})$  of  $\rho$  such that

$$\text{QCorr}(\rho) \geq \frac{k}{2k-2} \text{QCorr}(|\phi\rangle).$$

Thus, we could find a pure state  $|\theta\rangle \in \bigotimes_{i=1}^k \mathcal{H}_{A'_i}$  that makes  $F(|\phi\rangle\langle\phi|, |\phi'\rangle\langle\phi'|) \geq 1 - \epsilon$ , where  $|\phi'\rangle = |\psi\rangle \otimes |\theta\rangle$ . By the definition of  $\text{QCorr}_\epsilon^{\text{pure}}(|\phi'\rangle)$ , we have that

$$\text{QCorr}(|\phi\rangle) \geq \text{QCorr}_\epsilon^{\text{pure}}(|\phi'\rangle).$$

Combining the above two inequalities, we see that

$$\text{QCorr}(\rho) \geq \frac{k}{2k-2} \text{QCorr}_\epsilon^{\text{pure}}(|\phi'\rangle) \geq \frac{k}{2k-2} \text{M}_\epsilon(|\phi'\rangle),$$

where the last inequality comes from Theorem 9. According to Lemma 5.2 of [6], we have that  $\text{M}_\epsilon(|\phi'\rangle) \geq \text{M}_\epsilon(|\psi\rangle)$ . Applying Theorem 9 again, we eventually get that  $\text{QCorr}_\epsilon^{\text{pure}}(|\phi'\rangle) \geq \text{M}_\epsilon(|\psi\rangle) \geq \text{QCorr}_{k\epsilon}^{\text{pure}}(|\psi\rangle)$ . This means that

$$\text{QCorr}_\epsilon(|\psi\rangle) = \text{QCorr}(\rho) \geq \frac{k}{2k-2} \text{QCorr}_\epsilon^{\text{pure}}(|\phi'\rangle) \geq \frac{k}{2k-2} \text{QCorr}_{k\epsilon}^{\text{pure}}(|\psi\rangle),$$

and the proof is completed.  $\square$

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